Student Number ______________________

Ascham School

Mathematics Extension 1

Trial HSC Examination

Monday 29th July 2019
2 hours

General Instructions

- Reading time – 5 minutes.
- Working time – 2 hours.
- Write using black non-erasable pen.
- NESA-approved calculators may be used.
- A Reference Sheet is provided.
- In Questions 11–14, show relevant mathematical reasoning and/or calculations.

Total marks – 70

Section I Pages 3 – 6
10 marks

- Use the Multiple Choice Answer Sheet provided to answer Q1-10.
- Allow about 15 minutes for this section.

Section II Pages 7 – 13
60 marks

- Answer Questions 11-14.
- Answer each question in a new booklet.
- Label all sections clearly with your name/number and teacher's initials.
- Allow about 1 hour and 45 minutes for this section.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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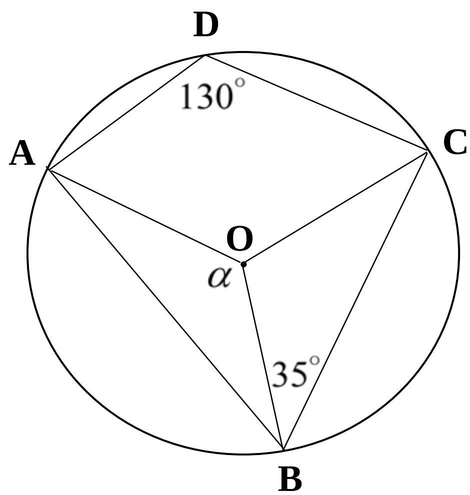
1 A, B, C and D are points on the circumference of the circle centre O.
$\angle OBC = 35^\circ$, $\angle ADC = 130^\circ$ and $\angle AOB = \alpha$. The size of α is:

(A) 70°

(B) 110°

(C) 150°

(D) 185°



NOT TO SCALE

2 Consider the function $f(x) = x^2 + 6x$.
Select the largest domain of $f(x)$ for which there exists an inverse function $f^{-1}(x)$.

(A) $x \geq 0$

(B) $x \geq -3$

(C) $x \geq -6$

(D) $x \geq -9$

3 Suppose $x^3 - 3x^2 + a \equiv (x-2)Q(x) + 1$, where $Q(x)$ is a polynomial.
The correct value of a is:

(A) -2

(B) 2

(C) 1

(D) 5

4 A primitive function of $\dfrac{1}{1+4x^2}$ is:

(A) $\dfrac{1}{4}\tan^{-1}(2x) + C$

(B) $\dfrac{1}{2}\tan^{-1}(2x) + C$

(C) $\dfrac{1}{4}\tan^{-1}\left(\dfrac{x}{2}\right) + C$

(D) $\dfrac{1}{2}\tan^{-1}\left(\dfrac{x}{2}\right) + C$

5 If $\dfrac{dP}{dt} = 0.2(P-10)$ and $P = 30$ when $t = 0$, which of the following is an expression for P?

(A) $P = 10 + 20e^{0.2t}$

(B) $P = 20 + 10e^{0.2t}$

(C) $P = 20 + 30e^{0.2t}$

(D) $P = 30 + 20e^{0.2t}$

6 The gradient of the tangent to the curve $y = \tan^{-1}(\sin x)$ at $x = 0$ is:

(A) undefined

(B) 0

(C) 1

(D) -1

7 The displacement x of a particle at time t is given by:

$$x = 5\sin 2t + 12\cos 2t\,.$$

What is the amplitude of the particle?

(A) 12

(B) 13

(C) 17

(D) 26

8 Choose the correct value of $\lim\limits_{x \to 0} \dfrac{\cos(2x) - 1}{x^2}$.

(A) -2

(B) -1

(C) 1

(D) 2

9 The rise and fall of the tide is assumed to be simple harmonic, with the time between low and high tide being six hours.

The water depth at a harbour entrance at high and low tides are 14 metres and 10 metres respectively.

If t is the number of hours after low tide, and y is the water depth in metres, which equation models this information?

(A) $y = 12 - 2\cos(6t)$

(B) $y = 12 - 2\cos\left(\dfrac{\pi t}{6}\right)$

(C) $y = 12 + 2\cos\left(\dfrac{\pi t}{6}\right)$

(D) $y = 12 + 2\cos(6t)$

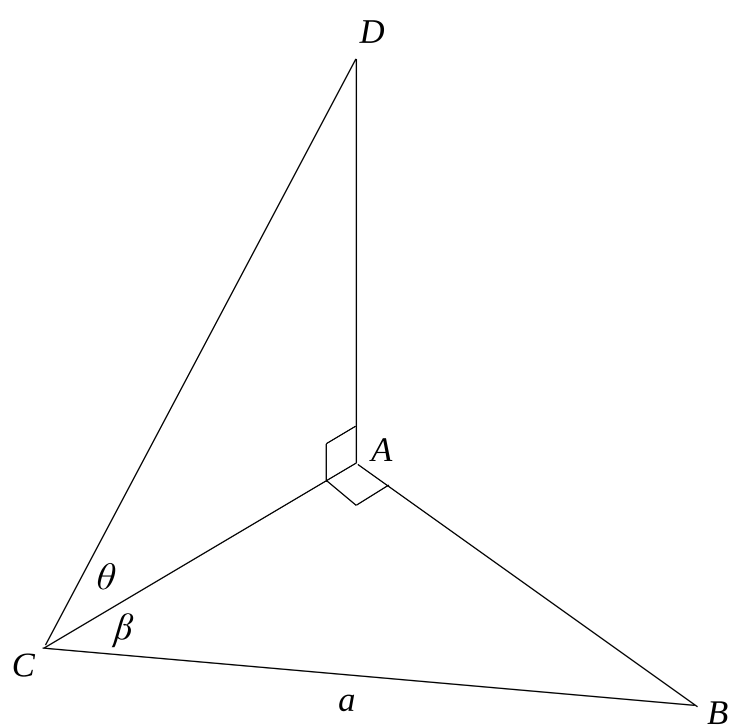
10 In the figure, ABC is a triangle on a horizontal plane and AD is a vertical flag pole. If $BC = a$, which of the following expressions is equal to AD?

(A) $a\sin(\beta + \theta)$

(B) $a\cos\beta\sin\theta$

(C) $a\sin\beta\tan\theta$

(D) $a\cos\beta\tan\theta$



Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.
In Questions 11–14, you should include relevant mathematical reasoning and/ or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

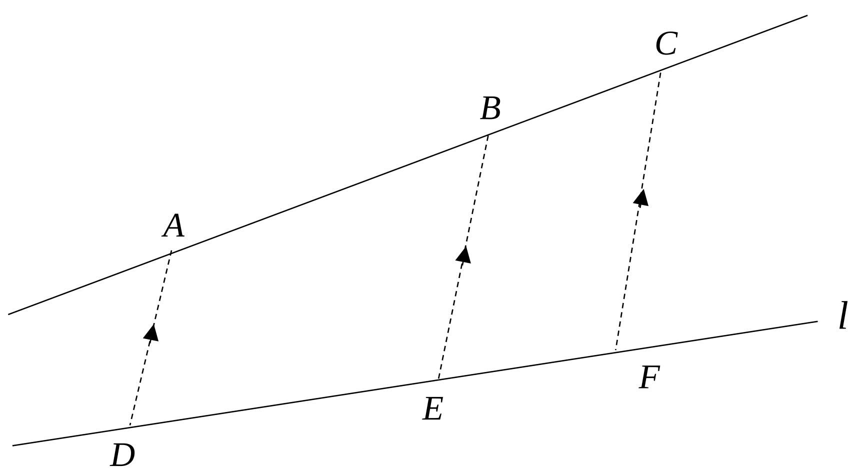
(a) Evaluate $\int_0^1 \frac{dx}{\sqrt{2-x^2}}$. **2**

(b) Solve $\frac{2}{3-x} \leq 1$. **3**

(c) A is the point $(-1, 1)$, B is the point $(3, 5)$ and C is the point (x, y).

(i) Find the coordinates of the point C which divides the interval joining AB **externally** in the ratio 3:1. **2**

(ii) Parallel lines are drawn through A, B and C and intersect with the line l at the points D, E and F respectively, as shown in the diagram.
What is the ratio of DE **:** EF ? **1**



NOT TO SCALE

Question 11 continues on the following page.

(d) An oil leak spreads on the ground in the shape of a circle. The radius of the circle increases from 0 cm, at a constant rate of 5 cm s^{-1}.
At what rate is the area of the circle increasing when the radius is 10 cm?
Express your answer in exact form. **3**

(e) The rate at which a body cools in air is proportional to the difference between its temperature (T) and the constant temperature of the surrounding air (S).

As a result, it follows that:

$$T = S + Be^{kt},$$

where t is the time in hours and B and k are constants.

A metal cake tin has a temperature of 180° C when removed from an oven.
In a room, the temperature of the surrounding air is 20° C.
The cake tin takes 10 minutes to cool to 60° C.

(i) Show that $k = \dfrac{\ln(0.25)}{10}$. **2**

(ii) Hence, find the time it takes for the cake tin to cool to 40° C when removed from the oven. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

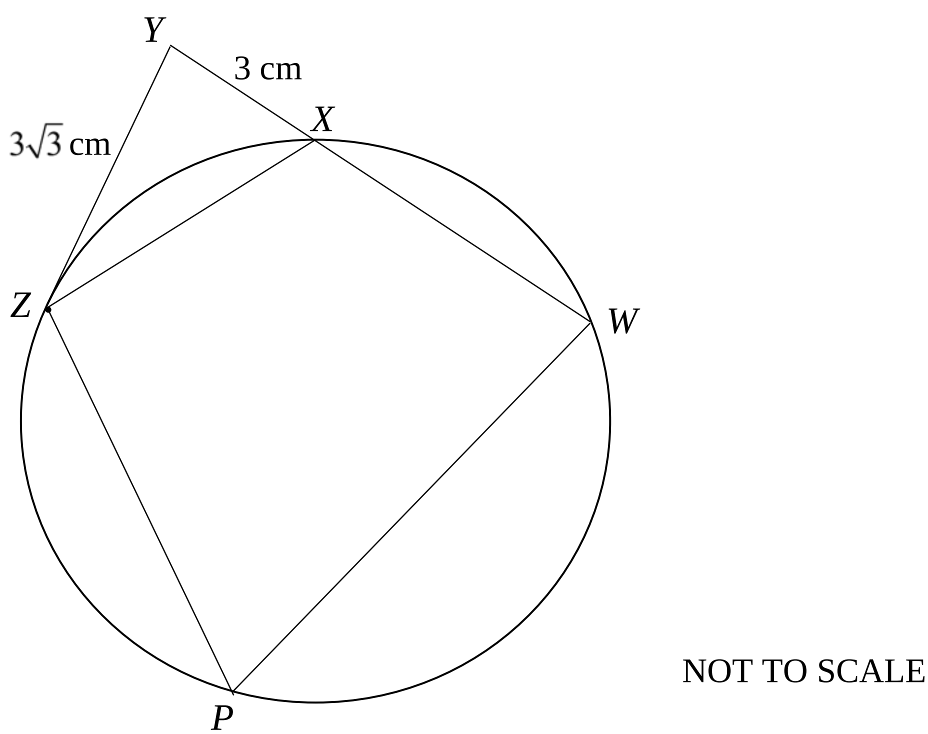
(a) Use the substitution $u = \sqrt{x}$ to find $\int \frac{dx}{x + \sqrt{x}}$, $x > 0$. **3**

(b) One of the roots of the equation $x^3 - kx^2 + 1 = 0$ is the sum of the other two roots.

(i) Show that $x = \frac{k}{2}$ is a root of the equation. **2**

(ii) Find the value of k. **1**

(c) In the diagram, the points P, W, X and Z lie on a circle and WX produced meets the tangent from Z at the point Y. It is known that $XY = 3$ cm and $YZ = 3\sqrt{3}$ cm.



Copy the diagram onto your paper.

(i) Find the length of XW, giving reasons. **2**

(ii) If XZ is the diameter of a smaller circle passing through X, Y and Z, find the size of $\angle WPZ$, giving reasons. **3**

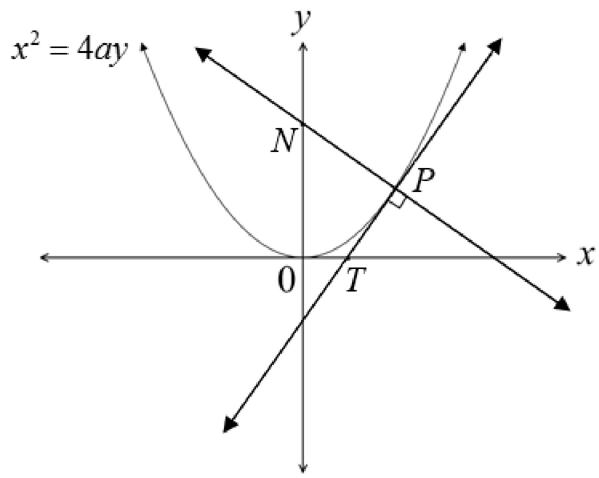
Question 12 continues on the following page.

(d) Consider the point $P(2ap,\ ap^2)$ which lies on the parabola $x^2 = 4ay$.
From the point P, the tangent to the parabola meets the x-axis at T.
The normal to the parabola at P meets the y-axis at N.

(i) With the help of the Reference Sheet, show that the coordinates of the points T and N are:

$$T\ (ap,\ 0) \quad \text{and} \quad N\ (0,\ ap^2 + 2a).$$ **2**

(ii) Find the locus of the midpoint M, of TN. **2**



NOT TO SCALE

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the equation $\tan^{-1} x - \dfrac{x}{2} = 0$.

Using $x = 2$ as the first approximation to one of the roots, use one application of Newton's method to find a better approximation to this root, correct to 2 decimal places. **3**

(b) Find the exact volume of the solid obtained by rotating $y = \cos^{-1} x$ about the y-axis between $y = 0$ and $y = \pi$. **3**

(c) Consider the function $f(x) = 2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1)$.

(i) Find the domain of $f(x)$. **1**

(ii) Show that $f'(x) = 0$. **2**

(iii) Sketch the graph of $y = f(x)$. **1**

(d) A particle moves in a straight line such that its velocity v m/s is given by $v = 2\sqrt{4x-1}$ when it is x metres from the origin. If $x = \dfrac{1}{4}$ when $t = 0$, find the:

(i) equation for $\ddot{x}$. **2**

(ii) equation for the displacement (x) in terms of time (t). **3**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) A particle moves in a straight line such that its acceleration $\ddot{x}$ m/s^2 is given by $\ddot{x} = 12 - 4x$ when it is x metres from the origin.

Initially, the particle is positioned at $x = 6$ with velocity $v = -2\sqrt{7}$ m/s.

(i) Explain why the particle moves in simple harmonic motion and state the value of n and the centre of motion. **2**

(ii) Hence show that $v^2 = 28 + 24x - 4x^2$. **2**

(iii) Over what range of x-values is the particle moving? **1**

(iv) Find the maximum speed of the particle. **1**

(b) Use the principle of mathematical induction to prove that $2^{3n} - 3^n$ is divisible by 5 for all integers $n \geq 1$. **3**

Question 14 continues on the following page.

(c) Jon Snow fires arrows with an initial velocity V m/s at an angle of 60° to the horizontal. The path of any arrow from the origin is given by the displacement functions below, where t is the time in seconds and g is the acceleration due to gravity in m/s^2:

$$x = \frac{Vt}{2}$$

(DO NOT PROVE THESE)

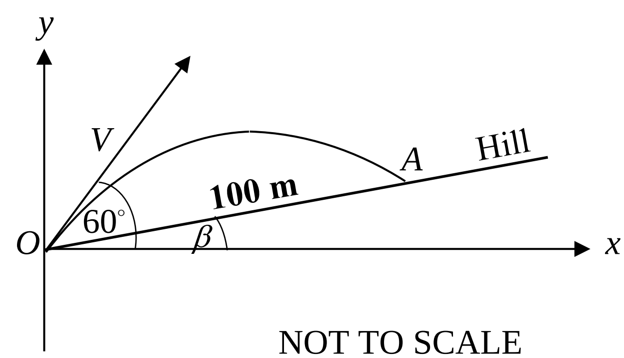
$$y = -\frac{1}{2}gt^2 + \frac{Vt\sqrt{3}}{2}$$

(i) Show that the Cartesian equation of an arrow's path is: $y = \sqrt{3}x - \dfrac{2gx^2}{V^2}$. **1**

(ii) Jon stood at the <u>bottom</u> of a hill inclined at an angle β to the horizontal. When he fired an arrow, it landed 100 metres <u>up</u> the hill at point A.

Use the result of part (i) to show that:

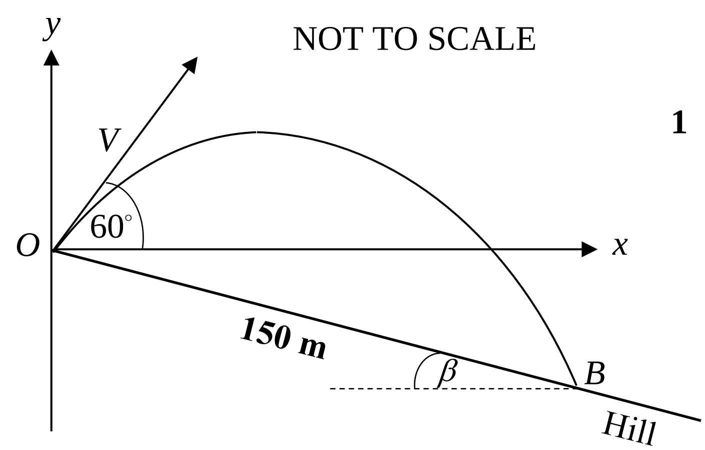
$$\tan\beta = \sqrt{3} - \frac{200g\cos\beta}{V^2}$$ **2**



(iii) Jon climbs the hill. Jon turns around and is now standing at the <u>top</u> of the same hill, looking down the hill. When he shoots an arrow at the same velocity and angle of projection, the arrow lands 150 metres <u>down</u> the hill, at point B.

Show that:

$$\tan\beta = \frac{300g\cos\beta}{V^2} - \sqrt{3}$$ **1**

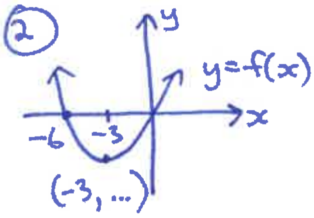


(iv) Hence, show that $\beta = \tan^{-1}\left(\dfrac{\sqrt{3}}{5}\right)$. **2**

End of Exam

Ascham Yr. 12 Mathematics Ext. 1 Trial Exam 2019

① Reflex $\angle AOC = 2 \times \angle ADC$
$= 260^\circ$
($\angle$ at centre is twice $\angle$ at circumference)
$\angle OCB = 35^\circ$ (base $\angle$s isosceles $\triangle$ equal)
$\angle BOC = 180^\circ - 2\times 35^\circ$ ($\angle$ sum of $\triangle$)
$= 110^\circ$
$\alpha = 260^\circ - 110^\circ$
$\therefore \alpha = 150$ $\quad\therefore$ C

②


$f(x) = x(x+6)$
Axis of symmetry is
$x = -3$
$\therefore$ Largest domain for $f^{-1}(x)$ is
$x \geq -3$ or $x \leq -3$ $\quad\therefore$ B

③ If $P(x) = x^3 - 3x^2 + a$,
then $P(2) = 1$, since
$P(x) = (x-2)Q(x) + 1$
Hence $2^3 - 3(2)^2 + a = 1$
$8 - 12 + a = 1$
$\therefore a = 5$ $\quad\therefore$ D

④ $\int \frac{1}{1+4x^2}\,dx$

$= \int \frac{1}{4(\frac{1}{4}+x^2)}\,dx$

$= \frac{1}{4}\int \frac{1}{(\frac{1}{2})^2 + x^2}\,dx$

$= \frac{1}{4} \times \frac{1}{1/2} \tan^{-1}\left(\frac{x}{1/2}\right) + C$

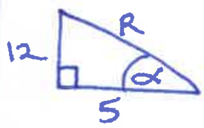
$= \frac{1}{2}\tan^{-1}(2x) + C$

$\therefore$ B

⑤ If $\frac{dP}{dt} = k(P-B)$, then
$P = B + Ae^{kt}$
Given $\frac{dP}{dt} = 0.2(P-10)$
then $P = 10 + Ae^{0.2t}$
$\therefore$ A
[since when $30 = 10 + Ae^0$,
then $A = 20$]

⑥ If $f(x) = \tan^{-1}(\sin x)$
$f'(x) = \frac{1}{1+\sin^2 x} \times \cos x$
Now $f'(0) = \frac{\cos(0)}{1+\sin^2(0)} = \frac{1}{1+0} = 1$
$\therefore$ C

⑦ Using $R\sin(2t+\alpha)$
$= R\sin(2t)\cos\alpha + R\cos(2t)\sin\alpha$
$R\cos\alpha = 5$ and $R\sin\alpha = 12$
$\cos\alpha = \frac{5}{R}$ $\quad \sin\alpha = \frac{12}{R}$



$R = \sqrt{5^2 + 12^2}$
$\therefore R = 13$ $\quad\therefore$ B

⑧ $\lim_{x\to 0} \frac{\cos(2x) - 1}{x^2}$

$= \lim_{x\to 0} \frac{1 - 2\sin^2 x - 1}{x^2}$

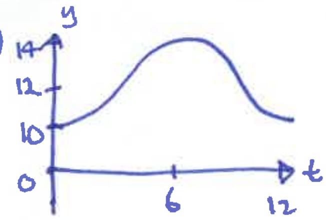
$= -2 \lim_{x\to 0}\frac{\sin x}{x} \lim_{x\to 0}\frac{\sin x}{x}$

$= -2 \times 1 \times 1$

$= -2$ $\quad\therefore$ A

(2)

(9)



amplitude = 2

Period = 12 hours

$\therefore \frac{2\pi}{n} = 12$

$\therefore n = \pi/6$

$y = b + a\cos(nt)$

$\Rightarrow y = 12 - 2\cos\left(\frac{\pi}{6}t\right) \quad \therefore \underline{B}$

(10) In $\triangle ABC$: $\cos\beta = \frac{AC}{a}$

$\therefore AC = a\cos\beta$

In $\triangle ACD$: $\tan\theta = \frac{AD}{AC}$

$\therefore \tan\theta = \frac{AD}{a\cos\beta}$

$\therefore AD = a\cos\beta\tan\theta \quad \therefore \underline{D}$

Question 11

a) $\int_0^1 \frac{1}{\sqrt{(\sqrt{2})^2 - x^2}}\,dx$

$= \left[\sin^{-1}\frac{x}{\sqrt{2}}\right]_0^1$ ✓

$= \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) - \sin^{-1}0$

$= \frac{\pi}{4} - 0$

$= \frac{\pi}{4}$ ✓

(2 marks)

b) $\frac{2}{3-x} \leq 1 \quad , \quad x \neq 3$

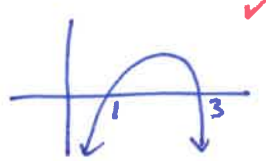
$\frac{2(3-x)^2}{3-x} \leq 1(3-x^2)$ ✓

$2(3-x) \leq (3-x)^2$

$2(3-x) - (3-x)^2 \leq 0$

$(3-x)[2-(3-x)] \leq 0$

$(3-x)(x-1) \leq 0$ ✓



$\therefore x \leq 1$ $\frac{1}{2}$

or

$x > 3$ $\frac{1}{2}$

(3 marks)

c) i) Let ratio be $m:n$

$= 3:-1$

If $(x_1, y_1) = (-1, 1)$

and $(x_2, y_2) = (3, 5)$

Use $\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$

$C = \left(\frac{3(3) - 1(-1)}{3+-1}, \frac{3(5) - 1(1)}{3+-1}\right)$

$C = (5, 7)$ (2 marks)

Deduct 1 mark for sign error

ii) Ratio of intercepts on parallel lines are equal

$\therefore DE : EF = AB : BC$

If C divides AB externally in the ratio 3:1, then

$AB : BC = 2:1$

$\therefore DE : EF = 2:1$ ✓

(1 mark)

d) $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$ ✓

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ ✓

$= 2\pi r \times 5$

$\therefore$ when $r = 10$,

$\frac{dA}{dt} = 100\pi \text{ cm}^2/\text{s}$ ✓

(3 marks)

e) i) $T = S + Be^{kt}$

$180^\circ = 20^\circ + Be^{k\times 0}$

$\therefore B = 160^\circ$ ✓

So $\boxed{T = 20 + 160e^{kt}}$

when $t = 10$, $T = 60$

$60 = 20 + 160e^{10k}$

$\frac{40}{160} = e^{10k}$ ✓

$\ln\left(\frac{1}{4}\right) = 10k$

$\therefore k = \frac{\ln(0.25)}{10}$

(2 marks)

ii) $40 = 20 + 160e^{kt}$

$\frac{40-20}{160} = e^{kt}$

$\frac{1}{8} = e^{kt}$ ✓

$\ln\left(\frac{1}{8}\right) = kt$

$t = \frac{\ln\left(\frac{1}{8}\right)}{k}$

$t = \frac{\ln\left(\frac{1}{8}\right)}{\left[\frac{\ln(0.25)}{10}\right]}$

$t = 15$ ✓

$\therefore$ It takes 15 minutes to cool to 40°C

(2 marks)

Question 12

a) If $u = \sqrt{x}$, $\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$

$du = \frac{dx}{2\sqrt{x}}$

$dx = 2\sqrt{x}\, du$

$\therefore dx = 2u\, du$ ✓

$\int \frac{dx}{x + \sqrt{x}}$

$= \int \frac{2u}{u^2 + u}\, du$

$= \int \frac{2u}{u(u+1)}\, du$

$= \int \frac{2}{u+1}\, du$ ✓

$= 2\ln(u+1) + c$

$= 2\ln(\sqrt{x}+1) + c$ ✓

(3 marks)

b) i) Let roots be $\alpha, \beta, \alpha+\beta$

sum of roots $= \frac{-b}{a}$

$\alpha + \beta + \alpha + \beta = \frac{-k}{1}$ ✓

$2(\alpha+\beta) = k$

$\alpha + \beta = \frac{k}{2}$

$\therefore$ one of the roots is $x = \frac{k}{2}$ ✓

(2 marks)

(ii) $\left(\frac{k}{2}\right)^3 - k\left(\frac{k}{2}\right)^2 + 1 = 0$

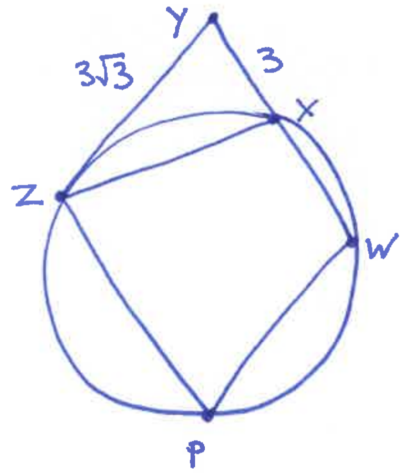
$\frac{k^3}{8} - \frac{k^3}{4} = -1$

$\frac{k^3 - 2k^3}{8} = -1$

$-k^3 = -8$

$\therefore k = 2$ ✓ (1 mark)

c)



(i) $YZ^2 = YW \times YX$

(square of tangent equals product of intercepts of secant) ✓

Let $XW = x$

$(3\sqrt{3})^2 = (3+x) \times 3$

$27 = 9 + 3x$

$x = 6$

$\therefore XW = 6\text{cm}$ ✓

(2 marks)

(ii) $\angle ZYX = 90°$

(∠ in a semi-circle equals 90°) ✓

In $\triangle XYZ$: $\tan \angle YXZ = \dfrac{3\sqrt{3}}{3}$

$\therefore \angle YXZ = 60°$ ✓

$\therefore \angle WPZ = 60°$

(exterior ∠ of cyclic quadrilateral equals interior opposite ∠) ✓

(3 marks)

d)

(i) Tangent at P: $y = px - ap^2$

At T, where $y=0$: $0 = px - ap^2$

$px = ap^2$

$x = \dfrac{ap^2}{p}$

$x = ap$ ✓

$\therefore$ T is $(ap, 0)$

(1 mark)

Normal at P: $x + py = ap^3 + 2ap$

At N, where $x=0$:

$0 + py = ap^3 + 2ap$

$y = \dfrac{ap^3 + 2ap}{p}$

$y = ap^2 + 2a$ ✓

$\therefore$ N is $(0, ap^2 + 2a)$

(1 mark)

(ii) Midpoint (M) of TN

$= \left(\dfrac{ap+0}{2}, \dfrac{0 + ap^2 + 2a}{2}\right)$

i.e. $x = \dfrac{ap}{2}$ and $y = \dfrac{ap^2 + 2a}{2}$ ...(2)

$\therefore p = \dfrac{2x}{a}$ ...(1) ✓

sub. eqn (1) into (2):

$y = \dfrac{a\left(\frac{2x}{a}\right)^2 + 2a}{2}$

$y = \dfrac{\frac{4x^2}{a} + 2a}{2}$

$\therefore y = \dfrac{2x^2}{a} + a$ ✓

(2 marks)

Question 13

a) If $f(x) = \tan^{-1}x - \frac{x}{2}$

then $f'(x) = \frac{1}{1+x^2} - \frac{1}{2}$ ✓

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Using $x_1 = 2$:

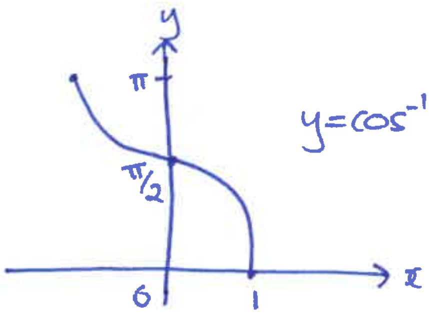
$$x_2 = 2 - \left[\frac{\tan^{-1}(2) - \frac{2}{2}}{\frac{1}{1+2^2} - \frac{1}{2}}\right]$$ ✓

$x_2 = 2.357....$

$\therefore x_2 = 2.36$ (2 d.p.) ✓

(3 marks)

b)



Making x the subject:

$x = \cos y$

$$V = 2\pi \int_0^{\pi/2} \cos^2 y \, dy$$ ✓

$$V = 2\pi \int_0^{\pi/2} \frac{1}{2}(\cos 2y + 1) \, dy$$

$$V = \pi \left[\frac{\sin 2y}{2} + y\right]_0^{\pi/2}$$ ✓

$$V = \pi\left[\frac{\sin(2\times\frac{\pi}{2})}{2} + \frac{\pi}{2} - \left(\frac{\sin(0)}{2} + 0\right)\right]$$

$$V = \pi\left[0 + \frac{\pi}{2} - 0\right]$$

$\therefore V = \frac{\pi^2}{2}$ units^3 ✓

(3 marks)

c) $f(x) = 2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1)$

i) $x \geq 0$ and $0 \leq \sqrt{x} \leq 1$ $\therefore 0 \leq x \leq 1$

and $-1 \leq 2x - 1 \leq 1$

$0 \leq 2x \leq 2$

$\therefore$ $0 \leq x \leq 1$ ✓

(1 mark)

ii)

$$f'(x) = \frac{2}{\sqrt{1-(\sqrt{x})^2}} \times \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{\sqrt{1-(2x-1)^2}} \times 2$$ ½ ½

$$= \frac{1}{\sqrt{x}\sqrt{1-x}} - \frac{2}{\sqrt{1-(4x^2-4x+1)}}$$

$$= \frac{1}{\sqrt{x}\sqrt{1-x}} - \frac{2}{\sqrt{4x(x-1)}}$$ ½

$$= \frac{1}{\sqrt{x}\sqrt{1-x}} - \frac{2}{2\sqrt{x}\sqrt{x-1}}$$ ½

$$= \frac{1}{\sqrt{x}\sqrt{1-x}} - \frac{1}{\sqrt{x}\sqrt{x-1}}$$

$$= 0$$

(2 marks)

iii) when $x = 0$:

$f(0) = 2\sin^{-1}\sqrt{0} - \sin^{-1}[2(0) - 1]$

$= 2\times 0 - \sin^{-1}(-1)$

$= 0 - (-\frac{\pi}{2})$

$= \frac{\pi}{2}$

(1 mark)

Deduct ½ mk for incorrect domain.

6

d) i) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$\ddot{x} = \frac{d}{dx}\left[\frac{1}{2}\left(2\sqrt{4x-1}\right)^2\right]$ ✓

$\ddot{x} = \frac{d}{dx}\left[2(4x-1)\right]$

$\therefore \ddot{x} = 8ms^{-2}$ ✓

(2 marks)

ii) $\frac{dx}{dt} = 2\sqrt{4x-1}$

$\frac{dt}{dx} = \frac{1}{2\sqrt{4x-1}}$ ½

$t = \int \frac{(4x-1)^{-\frac{1}{2}}}{2}\, dx$ ½

$t = \frac{(4x-1)^{\frac{1}{2}}}{2 \times \frac{1}{2} \times 4} + C$

$t = \frac{\sqrt{4x-1}}{4} + C$ ✓

when $t = 0$, $x = \frac{1}{4}$

$0 = \frac{\sqrt{4(\frac{1}{4})-1}}{4} + C$

$\therefore C = 0$ ½

$\therefore t = \frac{\sqrt{4x-1}}{4}$

$4t = \sqrt{4x-1}$

$16t^2 = 4x-1$

$4x = 16t^2+1$

$\therefore x = \frac{16t^2+1}{4}$ ½

(3 marks)

Question 14

a) i) $\ddot{x} = 12 - 4x$

$\ddot{x} = -4(x-3)$ ½

$\ddot{x} = -(2)^2(x-3)$

since $\ddot{x}$ is of the form $\ddot{x} = -n^2(x-b)$, ½ the particle moves in SHM where $n = 2$ ½ and centre is $x = 3$ ½ (2 marks)

ii) $\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$12 - 4x = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$ ½

$\int 12 - 4x \, dx = \frac{1}{2}v^2$

$12x - \frac{4x^2}{2} + C = \frac{1}{2}v^2$ ½

when $x = 6$ $v = -2\sqrt{7}$

$12(6) - 2(6)^2 + C = \frac{1}{2}(-2\sqrt{7})^2$

$72 - 72 + C = \frac{1}{2}(28)$

$\therefore C = 14$ ½

$\frac{1}{2}v^2 = 12x - 2x^2 + 14$ ½

$\therefore v^2 = 28 + 24x - 4x^2$

(as required) (2 marks)

iii) Solve $28 + 24x - 4x^2 = 0$

$4x^2 - 24x - 28 = 0$

$4(x^2 - 6x - 7) = 0$

$4(x-7)(x+1) = 0$

$\therefore x = 7$ or $x = -1$

so particle moves in the range $-1 \le x \le 7$ ✓ (1 mark)

iv) Max speed occurs at $x = 3$ (centre of motion)

$v^2 = 28 + 24(3) - 4(3)^2$

$v^2 = 64$

$\therefore$ max speed $= \sqrt{64} = 8$ m/s ✓ (1 mark)

b) Step ① Prove true for $n = 1$

$2^{3(1)} - 3^1 = 8 - 3$
$= 5$ ✓

which is divisible by 5

$\therefore$ it is true for $n = 1$

Step ② Assume it is true for $n = k$

i.e. $2^{3k} - 3^k = 5M$ $(M \in \mathbb{Z})$ ✓

$\therefore$ $\boxed{2^{3k} = 5M + 3^k}$ $*$

Step ③ Prove it is true for $n = k+1$

i.e. RTP $2^{3(k+1)} - 3^{k+1} = 5Q$ $(Q \in \mathbb{Z})$

LHS $= 2^{3k+3} - 3^{k+1}$

$= 2^{3k} \times 2^3 - 3^k \times 3^1$

$= (5M + 3^k)8 - 3^k \times 3$ ...using $*$

$= 40M + 8(3^k) - 3(3^k)$

$= 40M + 5(3^k)$

$= 5(8M + 3^k)$ ✓

which is divisible by 5

Hence it is proved true by Mathematical Induction. (3 marks)

Alternate Solution:

Step ② $2^{3k} - 3^k = 5M$

Step ③ LHS $= 2^{3k} \times 2^3 - 3 \times 3^k$

$= 2^{3k}(8) - 3(2^{3k} - 5M)$

$= 8(2^{3k}) - 3(2^{3k}) + 15M$

$= 5(2^{3k}) + 15M$

$= 5(2^{3k} + 3M)$

which is divisible by 5

c) i) From $x=\frac{Vt}{2}$, $t=\frac{2x}{V}$

sub. into $y=-\frac{1}{2}gt^2+\frac{Vt\sqrt{3}}{2}$

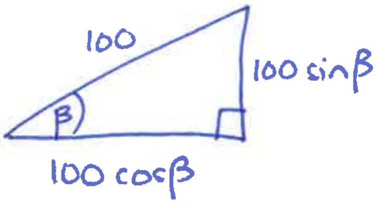
$$\Rightarrow y=-\frac{1}{2}g\left(\frac{2x}{V}\right)^2+V\left(\frac{2x}{V}\right)\frac{\sqrt{3}}{2}$$

$$y=-\frac{2gx^2}{V^2}+x\sqrt{3}$$

i.e. $y=\sqrt{3}x-\frac{2gx^2}{V^2}$

(as required) (1 mark)

ii)



sub. $x=100\cos\beta$

and $y=100\sin\beta$ into

$$y=\sqrt{3}x-\frac{2gx^2}{V^2}$$

$$100\sin\beta=\sqrt{3}(100\cos\beta)-\frac{2g(100\cos\beta)^2}{V^2}$$

$$100\sin\beta=100\sqrt{3}\cos\beta-\frac{20000g\cos^2\beta}{V^2}$$

Divide both sides by $100\cos\theta$:

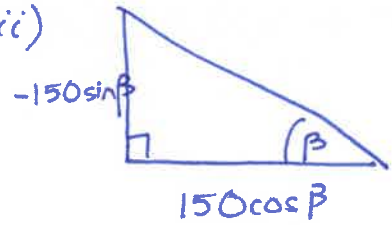
$$\frac{100\sin\beta}{100\cos\beta}=\frac{100\sqrt{3}\cos\beta}{100\cos\beta}-\frac{20000g\cos^2\beta}{V^2\,100\cos\beta}$$

$$\therefore \tan\beta=\sqrt{3}-\frac{200g\cos\theta}{V^2} \quad \ldots (1)$$

(as required)

(2 marks)

iii)



sub. $x=15\cos\beta$

and $y=-150\sin\beta$ into

$$y=\sqrt{3}x-\frac{2gx^2}{V^2}$$

$$-150\sin\beta=\sqrt{3}(150\cos\beta)-\frac{2g(150\cos\beta)^2}{V^2}$$

$$-150\sin\beta=150\sqrt{3}\cos\beta-\frac{45000g\cos^2\beta}{V^2}$$

Divide both sides by $150\cos\beta$:

$$\frac{-150\sin\beta}{150\cos\beta}=\frac{150\sqrt{3}\cos\beta}{150\cos\beta}-\frac{45000g\cos^2\beta}{V^2\,150\cos\beta}$$

$$-\tan\beta=\sqrt{3}-\frac{300g\cos\beta}{V^2}$$

i.e. $\tan\beta=\frac{300g\cos\beta}{V^2}-\sqrt{3} \quad \ldots (2)$

(as required) (1 mark)

iv)

(1) ×3: $3\tan\beta=3\sqrt{3}-\frac{600g\cos\beta}{V^2} \quad \ldots (3)$

(2) ×2: $2\tan\beta=\frac{600g\cos\beta}{V^2}-2\sqrt{3} \quad \ldots (4)$

(3) + (4): $5\tan\beta=3\sqrt{3}+-2\sqrt{3}$

$$5\tan\beta=\sqrt{3}$$

$$\tan\beta=\frac{\sqrt{3}}{5}$$

$$\therefore \beta=\tan^{-1}\left(\frac{\sqrt{3}}{5}\right)$$

(as required) (2 marks)

[Award 1 mk for some progress in right direction]

P.T.O. for alternate solution...

END OF EXAM

Alternate solution to iii)

Rearranging eqn ①:

$$\frac{g\cos\beta}{v^2} = \frac{\tan\beta - \sqrt{3}}{-200}$$

sub. into eqn ②:

$$\tan\beta = 300\left(\frac{\tan\beta - \sqrt{3}}{-200}\right) - \sqrt{3}$$

$$200\tan\beta = -300\left(\tan\beta - \sqrt{3}\right) - 200\sqrt{3}$$

$$200\tan\beta = -300\tan\beta + 300\sqrt{3} - 200\sqrt{3}$$

$$500\tan\beta = 100\sqrt{3}$$

$$\tan\beta = \frac{100\sqrt{3}}{500}$$

$$\tan\beta = \frac{\sqrt{3}}{5}$$

$$\therefore \quad \beta = \tan^{-1}\left(\frac{\sqrt{3}}{5}\right)$$